ELASTIC CONSTANTS - THEIR DYNAMIC MEASUREMENT AND CALCULATION

ELASTISCHE PARAMETER - IHRE DYNAMISCHE MESSUNG UND BERECHNUNG

PARAMETRES ELASTIQUES - MESURE DYNAMIQUE ET CALCUL

Bernd Weiler, Christian U. Grosse

SUMMARY

The modulus of elasticity and the shear modulus can be calculated from the free vibrations of bar-shaped specimens. Therefore, a review of the theoretical bases for the calculation of the moduli is given. Measurements using the *Grindo-Sonic* device and a device developed at the FMPA were carried out and compared. Both devices turned out to be very efficient, although their features are quite different.

ZUSAMMENFASSUNG

Elastizitäts- und Schermoduln können aus den Eigenschwingungen von stabförmigen Probekörpern berechnet werden. Dazu wird eine Übersicht über die theoretischen Grundlagen der Berechnung gegeben. Mit dem *Grindo-Sonic*-Gerät und mit einer an der FMPA entwickelten Anlage wurden Messungen gemacht und verglichen. Beide Geräte erwiesen sich als sehr leistungsfähig, obwohl ihre Eigenschaften recht verschieden sind.

RÉSUMÉ

Les modules d'élasticité et de rigidité peuvent être calculés à partir des vibrations libres issues des specimens en forme de barre. Dans ce but, on a présenté un survol des bases pour le calcul théorique de ces modules. Au FMPA, un nouveau dispositif a été développé permettant la mesure de ces modules. Ce dispositif a été comparé au dispositif *Grindo-Sonic*. Les deux dispositifs ont permis une évaluation très satisfaisante des modules, bien qu'ils présentent des caracteristiques très differentes.

KEYWORDS: modulus of elasticity, shear modulus, resonance

NOTATIONS

E	modulus of elasticity (Young's modulus)
G	shear modulus (modulus of rigidity)
ν	Poisson's ratio
f_{tors}, f_{long}, f_{flex}	resonant frequency of torsional, longitudinal, flexural vibration
k	harmonic order (k=1 for the fundamental mode)
m	mass of the specimen
l	length of the specimen
b	width
h	height (direction of vibration)
r	radius
A	cross-section area
I	moment of inertia of cross-section
R, C, T	correction factors
v_p, v_s	compression wave velocity, shear wave velocity

1. INTRODUCTION

In contrast to the compressive strength which is usually determined to describe the behaviour of concrete specimens under loading conditions, the elastic parameters of concrete are physical constants. As a consequence, the elastic parameters can be derived from measurements of various physical properties of the specimen, such as wave velocities or resonance frequencies. These properties can be measured by non-destructive methods which enables the comparison of values determined by different testing methods applied on the same specimen.

The present work uses a method called resonant-frequency method [RILEM NDT-2, 1984] basing on the measurement of the natural frequencies of different vibration modes of concrete specimens. After calculating the modulus of elasticity and the shear modulus, the ultrasonic wave velocities can be derived. These velocities can be compared to the velocities measured directly by travel time measurements on the same specimens.

Due to thermal effects, the elastic constants measured as mentioned above are different from those measured under static conditions. The so-called static constants are commonly determined from the slope of the stress-strain diagram of specimens under load. No general analytical correlation can be found but, as a rule of thumb, the dynamic elastic constants of concrete are by 10 % larger than the corresponding static constants; for detailed calculations see [ROST and MONECKE, 1988]. For this article, dealing with the dynamic constants, the designation dynamic will be omitted.

Within the framework of the collaborative research center SFB 381 of the Deutsche Forschungsgemeinschaft DFG, the aim of this work was to provide reliable values of the elastic constants for a forward-modelling of ultrasonic wave propagation in concrete. Additionally, it gave the opportunity to test the reliability of the black-box testing device *Grindo-Sonic*, which is a popular tool in engineering.

2. PHYSICAL BASES

2.1 General relations

The relations between stresses, strains, elastic constants and wave velocities in elastic materials have already been studied elsewhere [SCHREIBER et al., 1973, GROSSE and REINHARDT, 1993]. Here a brief summary:

The general mass law for perfectly elastic materials can be expressed by a linear tensor function of the stress tensor T_{ik} and the deformation tensor D_{lm}, related by the elastic stiffness tensor C_{iklm}, which has 81 matrix elements:

$$T_{ik} = C_{iklm} D_{lm}$$

For isotropic materials, this tensor is reduced to 2 elastic constants. In continuum mechanics, usually the Lamé's constants λ and μ are used:

$$C_{iklm} = \lambda \delta_{ik} \delta_{lm} + \mu \left(\delta_{il} \delta_{km} + \delta_{im} \delta_{kl} \right)$$

Other sets of constants are frequently used in experimental mechanics, such as:

1. Poisson's ratio ν and bulk modulus of elasticity κ, where

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \quad \text{and} \quad \kappa = \lambda + \frac{2}{3}\mu .$$

2. modulus of elasticity E and shear modulus G, where

$$E = \frac{3\lambda + 2\mu}{\lambda + \mu}\mu \quad \text{and} \quad G = \mu .$$

Hence the following two relations for the compression wave velocity and the shear wave velocity hold:

$$v_p = \sqrt{\frac{G}{\rho}\frac{4G-E}{3G-E}} \qquad \text{and} \qquad v_s = \sqrt{\frac{G}{\rho}} \qquad \text{resp.} \tag{1}$$

2.2 Vibration modes and elastic constants

After exciting a structure by a mechanical impact, it starts to oscillate in the natural frequencies of its different vibration modes, according to the type of excitation. From an other point of view, these vibrations can be interpreted as standing waves. The relations between the resonant frequencies and the properties of the test specimens for simple geometries and homogeneous materials are a solved problem of continuum mechanics and can be found in general literature of both experimental and theoretical physics [e. g. BERGMANN and SCHAEFER, 1974]. These formulas are approximations for bar-like specimens of simple cross-sectional shapes, where the cross dimensions are small against the length of the specimen. When the ratio of length to cross dimensions is small, correction factors (*R, C, T,* resp.) must be added to the formulas. For ratios of down to 2, the following formulas can be applied [RILEM NDT-2, 1984, ASTM C215-91, 1991]:

Torsional vibrations:

$$G = \frac{f_{tors}^{\;2}}{k^2} \cdot \frac{ml}{A} \cdot 4 \cdot R \tag{2}$$

where

$$R = 1$$

for specimens with circular cross-section, and

$$R = \frac{b/h + h/b}{4\,b/h - 2{,}52(b/h)^2 + 0{,}21(b/h)^6}$$

for specimens with rectangular cross-section.

Longitudinal vibrations:

$$E = \frac{f_{long}^{\;2}}{k^2} \cdot \frac{ml}{A} \cdot 4 \cdot C \tag{3}$$

where

$$C = 1 + \frac{k^2 \pi^2 \nu^2 I}{Al^2}.$$

The moment of inertia of cross-section is calculated as

$$I = \frac{\pi r^4}{4}$$

for specimens with circular cross-section, and

$$I = \frac{bh^3}{12}$$

for specimens with rectangular cross-section.

Flexural vibrations:

$$E = \frac{f_{flex}^{\;2}}{(2k+1)^4} \cdot \frac{ml^3}{I} \cdot \frac{64}{\pi^2} \cdot T \tag{4}$$

where the correction factor is [MARTINCEK, 1962]:

$$T = \frac{1}{2} + \frac{(2n+1)^2}{8}\pi^2 \frac{I}{Al^2}\left[1+\frac{2(1+\nu)}{k}\right]$$
$$+\sqrt{\frac{1}{4} + \frac{(2n+1)^2}{8}\pi^2 \frac{I}{Al^2}\left[1+\frac{2(1+\nu)}{k}\right] + \frac{(2n+1)^4}{64}\pi^4 \frac{I^2}{A^2 l^4}\left[1-\frac{2(1+\nu)}{k}\right]^2}\,.$$

This correction factor is a function of the geometric dimensions of the test specimen given by $q = r/2l$ for cylindrical, and $q = h/l\sqrt{12}$ for prismatic specimens. T as a function of q (n=1) is plotted in fig. 1 for different values of the Poisson's ratio.

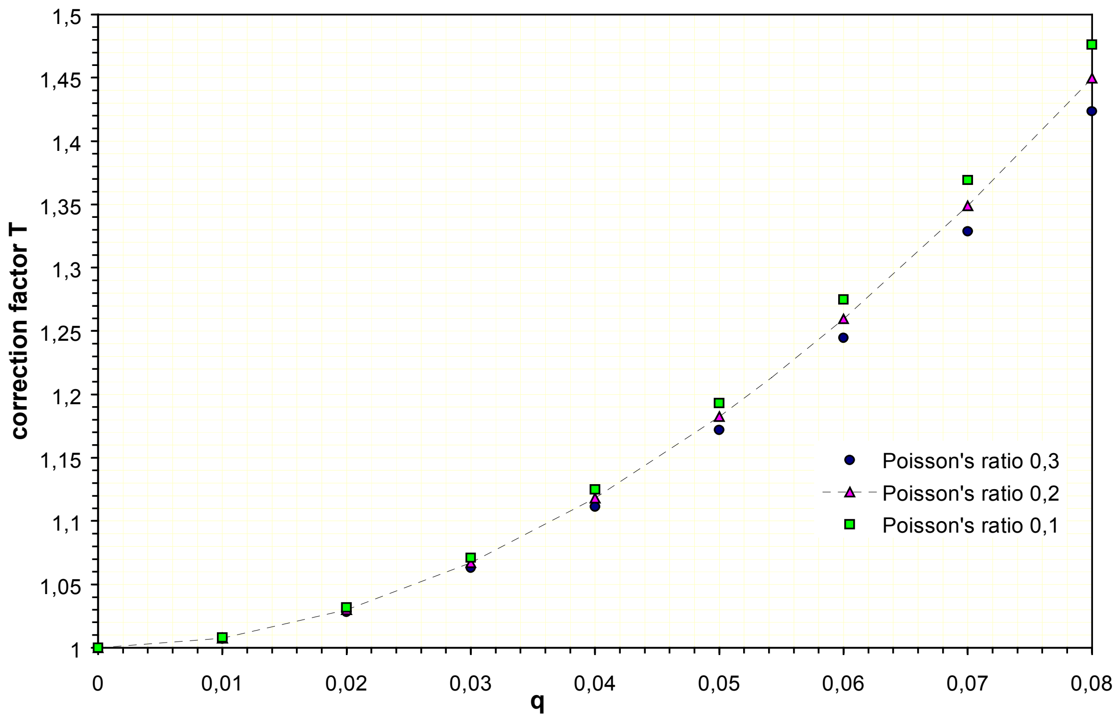

Fig. 1: *Correction factor T as a function of q*

Accepting an error of 1%, the correction factor *C* for longitudinal vibrations can be neglected, when the length of the specimen is at least three times the cross dimension. For the correction factor *T* of the flexural vibration, this ratio must exceed a value of 20. Due to the nature of the torsional vibration, the correction factor R may not be neglected in any case.

Calculating the modulus of elasticity from the flexural frequencies, these formulas show a high sensitivity concerning the measurement of the dimensions of the specimen. Compared with that, this modulus can be calculated rather accurate from the longitudinal frequencies, as well as the shear modulus, which shows exactly the same dependency on the torsional frequencies. In addition it can be realized that the frequencies of the harmonics are integer multiples of the fundamental frequency for longitudinal and torsional vibrations. The harmonics of flexural vibrations show an other behaviour and rise with a factor $(2k+1)^2$ of the fundamental.

3. EXPERIMENTAL

The specimens must be supported in such a way, that they can vibrate freely in the type and mode of vibration that has to be investigated. This can be achieved by supporting the specimens exactly in their nodal planes. The positions of the extreme vibration nodes, e. g., can be found at $0{,}224l$, $0{,}132l$, $0{,}094l$, $0{,}074l$ from the beam ends for the fundamental to the 3rd harmonic. To avoid the variation of the supports when testing the different modes, a simple foam rubber support was used, although then a higher damping of the vibrations had to be accepted.

The vibration was excited by dropping a small steel ball or by hitting the specimen with a steel rod. Another possibility is the use of an impulse force hammer for excitation. This provides the possibility of a deconvolution of the measured signal. The generated mechanical vibrations are transformed by an accelerometer into electrical oscillations.

The following figure shows the test setup with the positions of transducers and impactors as well as the nodal planes and the direction of movement for torsional, longitudinal, and flexural vibration, respectively.

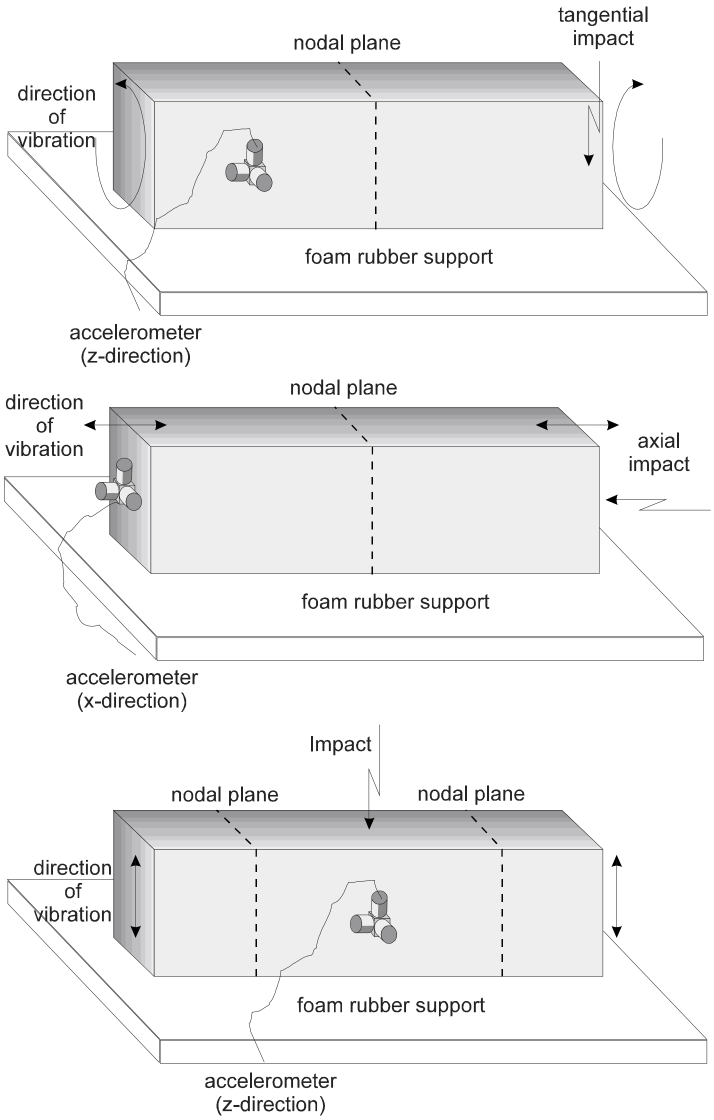

Fig. 2: *Test setup for torsional, longitudinal, and flexural vibrations (from top to bottom)*

The *Grindo-Sonic* equipment [LEMMONS GMBH] uses an piezoelectric unit that has to be held against the specimen by hand. After digital signal processing, the fundamental frequency is determined by counting zero crossings in the recorded waveform. Finally, the double period is displayed in units of microseconds (so-called “R-value“).

A more sophisticated equipment was used to control the results obtained by the *Grindo-Sonic* device. It consists of an 486 personal computer with an ADC-plugin board that operates at a sampling rate of 5 Mhz at 12 bit resolution. The signals were detected using a high sensitive piezoelectric accelerometer. This triaxial lightweight accelerometer (m=2,5 g) was coupled directly upon the specimen at the positions of maximum amplitude. As the transducer was small in the mass as compared with that of the specimens, its influence to the free vibration of the specimen could be neglected. Particular attention should be paid to the coupling of the transducer, that has to be proper and reproducible. Good results were obtained by means of a thin layer of wax. The recorded waveforms were transformed to frequency domain using a fast fourier transform; thus the whole spectrum could be analysed with care.

For a simple evaluation of the results concerning the elastic parameters according to the formulas (1) to (4), an evaluation worksheet basing on Microsoft® Excel was implemented, where both the parameters and the velocities are calculated automatically.

4. RESULTS

Measurements were carried out on different concrete prisms; flexural and torsional mode were chosen because of their clear vibration forms. Impacts and transducers were positioned according to figure 2. Tests perfomed with the *Grindo-Sonic* device show a great reproducibility (statistical error approximately 1 ‰). No difference could be realized between the excitation by means of bar or rod. A typical test series of ten measurements on a concrete prism of PZ 35 F (CEM I) with aggregates 0 - 8 mm is shown in table 1.

concrete prism measurement 18.4.95

length:	530,0 ± 0,5 mm	density:	2226 ± 25 kg/m³
width:	99,5 ± 0,5 mm	T-factor:	1,24
height:	100,0 ± 0,5 mm	R-factor:	1,1842
mass:	11,741 ± 0,001 kg		

test no.	1	2	3	4	5	6	7	8	9	10	average	s.d.
R-value	1537,2	1535,8	1538,9	1539,2	1536,6	1539,8	1538,0	1537,2	1537,4	1537,9	1537,8	
f_{flex} [Hz]	1301,1	1302,3	1299,6	1299,4	1301,6	1298,9	1300,4	1301,1	1300,9	1300,5	1300,6	1,0
test no.	1	2	3	4	5	6	7	8	9	10	average	s.d.
R-value	904,0	902,1	901,0	900,8	900,5	901,3	900,5	901,4	901,2	900,2	901,3	
f_{tor} [Hz]	2212,4	2217,0	2219,8	2220,2	2221,0	2219,0	2221,0	2218,8	2219,3	2221,7	2219,0	2,6

modulus of elasticity *E*:	**35256 ±862**	**N/mm²**
shear modulus *G*:	**14587 ±195**	**N/mm²**
v_D (calculated):	**4218 ±55**	**m/s**
v_S (calculated):	**2560 ±5**	**m/s**

Tab. 1: *Test carried out with the Grindo-Sonic Device*

Analogous measurements were carried out on the same specimens using the equipment designed at the FMPA. The results according to the previous *Grindo-Sonic* test are shown in table 2. The ADC-board was working at a sampling rate of 100 kHz recording 32 ksamples, what corresponds to a resolution of 3 Hz in the frequency domain. This resolution limit at the same time turned out to be the maximum error in determining the resonance frequencies, as was confirmed by all the measurements. Hence, the accuracy of the FMPA device is comparable to that of the *Grindo-Sonic* device, or even better.

concrete prism measurement 19.4.95

length:	530,0 ± 0,5 mm	density:	2226 ± 25 kg/m³	a/b:	0,995
width:	99,5 ± 0,5 mm	T-factor:	1,21	h/lsqr12	0,05447
height:	100,0 ± 0,5 mm	R-factor:	1,1842		
mass:	11,741 ± 0,001 kg				

test no.	1	2	3	4	5	6	7	8	9	10
f_{flex} [Hz]	1303	1303	1303	1303	1303	1303	1303	1303	1303	1303
test no.	1	2	3	4	5	6	7	8	9	10
f_{tor} [Hz]	2222	2222	2222	2222	2222	2222	2222	2222	2222	2222

modulus of elasticity *E*:	**34768 ±798**	**N/mm²**
shear modulus *G*:	**14626 ±162**	**N/mm²**
v_D (calculated):	**4137 ±40**	**m/s**
v_S (calculated):	**2563 ±2**	**m/s**

Tab. 2: *Test carried out with the FMPA Device*

The frequency spectra for both types of vibration of this concrete specimen are displayed in figure 3 in order to give an idea of this kind of evaluation.

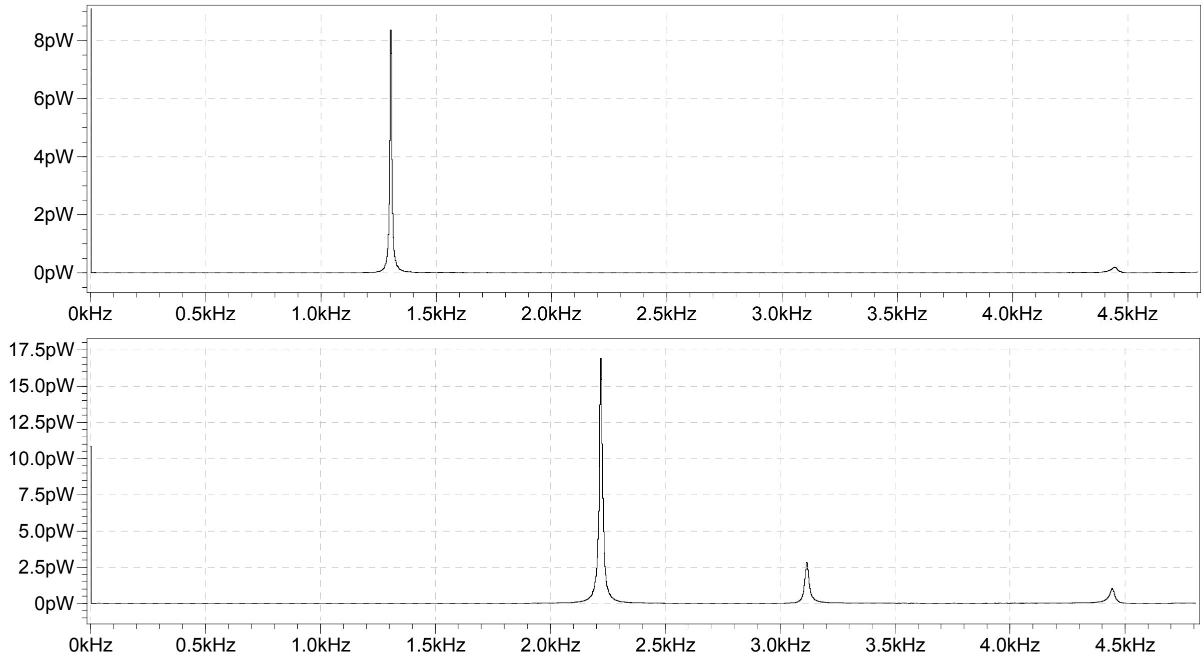

Fig. 3: *Flexural (top) and torsional (bottom) vibration spectra*

The flexural spectrum shows the strong fundamental flexural peak at ~1300 Hz and a very small peak at ~4450 Hz according to the first harmonic of the torsional mode. In the torsional spectrum, the strong fundamental torsional peak at ~2200 Hz is followed by the first harmonic of the flexural mode at ~3100 Hz and the first torsional harmonic at~4450 Hz.

To investigate this mixture of vibration types and modes, the transducer and the position of the excitation was varied freely over the whole specimen. A particularly interesting example of this series is shown in figure 4. Modes up to the fourth flexural and the sixth torsional harmonic could be identified. The identification of the modes was made easier by a calculation of the harmonics where the measured fundamental frequency is taken for exact (see table 3).

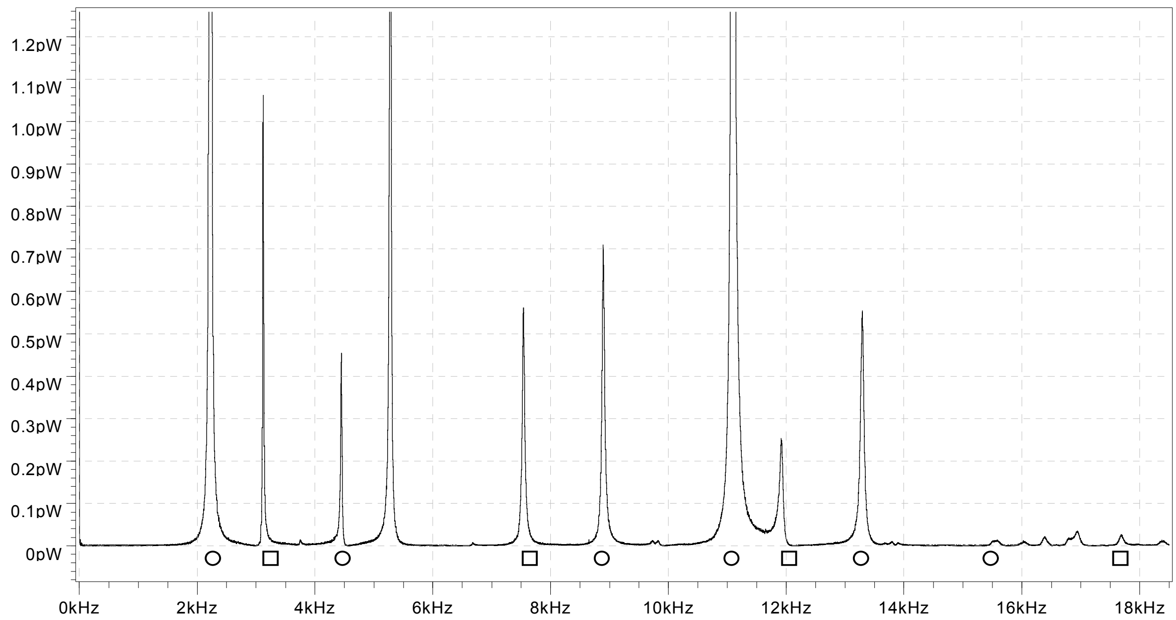

Fig. 4: *Frequency spectrum, torsionals marked by a circle O, flexurals by a square □*

	flexural		torsional	
	measured	calculated	measured	calculated
fundamental	1303 Hz	as measured	2222 Hz	as measured
1. harmonic	3116 Hz	3619 Hz	4444 Hz	4444 Hz
2. harmonic	7538 Hz	7094 Hz	≈ 6674 Hz	6666 Hz
3. harmonic	11914 Hz	11727 Hz	8893 Hz	8888 Hz
4. harmonic	≈ 17685 Hz	17518 Hz	11090 Hz	11110 Hz
5. harmonic			13285 Hz	13332 Hz
6. harmonic			15698 Hz	15554 Hz

Tab. 3: *Measured and calculated resonant frequencies according to fig. 4*

Additionally, the compression wave velocity of the concrete prism was measured directly to have another possibility for comparison. This was done using our equipment at maximum sampling rate and provided with a broadband transducer. We could measure

$$v_p = (4237 \pm 49)\ \text{m/s}$$

as statistical average of 20 tests with travel paths all over the specimen.

5. DISCUSSION AND CONCLUSIONS

The calculated moduli of

$$E = 35300 \pm 800 \text{ N/mm}^2 \text{ and } G = 14600 \pm 200 \text{ N/mm}^2$$

lie within the range of values found in literature about this test method [VINKELOE, 1962]. The calculated moduli can be determined with an accuracy of below 2,5% and 1,5 %, resp. As the calculated velocities become independent of the cross dimensions and masses, their error becomes smaller than 0,5 % for v_s and 1,5 % for v_p (which is, however, depending on the height). The directly measured value of v_p is a good confirmation of this fact and establishes this method as an alternative way of measuring wave velocities.

Although the calculation formulas for the elastic constants can be applied to specimens of different shapes and dimensions, comparisons of results obtained from specimens with far different characteristics should be made with caution. Numerous tests with both test systems revealed the problems associated with measuring the resonance frequencies of bar-like specimens. First of all, much care has to be taken in positioning transducers and impacts. This is a basic pre-condition for exciting and detecting the desired type and mode of vibration. Otherwise, a mixture of many vibration modes is created, which makes impossible a proper identification of the single modes and thus any evaluation of the test. As tests with the FMPA device showed, despite all care, always harmonics and other vibration types are being excited (see fig. 3) due to mode conversion in inhomogeneous areas. What is more, the two different ways of excitation are equivalent for the frequency range in question. In general, the impact time which is a function of hardness and size of the impactor determines the frequency range.

All these effects could only be studied and controlled by means of the FMPA system. Thus, for tests and operators with a scientific background, this is the system of first choice, as it provides control of all parameters that are influencing the tests.

The comparison between the *Grindo-Sonic* device and the FMPA system showed, within the limits of error, a good agreement in the determined resonance frequencies. This is a confirmation of the quality of the *Grindo-Sonic* system. It facilitates simple and fast testing due to the fact that no coupling of the transducer and no additional signal analysing is required. No control whatever is possible to check the results. In case of uncertainties regarding the measured parameters, this device is not able to eliminate errors caused by the operator. Hence, it is a good tool for quality control, where variations from well-known values of the elastic parameters have to be measured. On the other hand, is not suitable for measurements at materials where the elastic moduli are unknown.

Parameters that were not considered in this work are the temperature [SCHREIBER et al., 1973] and the moisture [ASTM C215-91, 1991] of the specimens.

ACKNOWLEDGEMENTS

The research described in this paper was supported by the Deutsche Forschungsgemeinschaft DFG (SFB 381). The authors would like to thank Prof. Dr.-Ing. H.-W. Reinhardt for thoughtful suggestions and Mr. T. Cramer for valuable calculations.

REFERENCES

ASTM C215-91 (1991): *Standard test method fundamental transverse, longitudinal, and torsional frequencies of concrete specimens*, 1994 Annual book of ASTM standards, Vol. 04.02, ASTM Philadelphia, Pa., 120 - 125

BERGMANN, L., SCHAEFER, C. (1974): *Lehrbuch der Experimentalphysik Vol. I: Mechanik, Akustik, Wärme 9th. edition*, W. de Gruyter, Berlin, 511 - 517

GROSSE, C., REINHARDT, H. (1993): *Basis for the determination of the elastic parameters in concrete referring to measurings of body and surface wave velocities*, Otto-Graf-Journal Vol. 4, 132 - 159

LEMMONS GMBH: *User manual for Grindo-Sonic Mk*, J. W. Lemmens GmbH, Köln

MARTINCEK, G. (1962): *Nedestructivne dynamicke metody skusania stavehnych materialov*, SAV, Bratislava

RILEM NDT-2 (1984): *Recommendations for the use of resonant-frequency method in testing concrete specimens 2nd edition*, RILEM recommendations NDT-2, E & FN Spon, London

ROST, L., MONECKE, J. (1988): *Nachwirkungsbedingte Unterschiede statischer und dynamischer Elastizitätsmoduln von Gesteinen*, Neue Bergbautechnik, Vol. 18, No. 1, 29 - 32

SCHREIBER, E., ANDERSON, O., SOGA, N. (1973): *Elastic constants and their measurement*, McGraw-Hill, New York, 1 - 8, 82 - 125

VINKELOE, R. (1962): *Prüfverfahren zur Ermittlung des dynamischen Elastizitätsmoduls von Betonprismen*, TfZ-Zbl. Vol. 86, No. 10, 272 - 276